Introduction to Acoustics and Aeroacoustical Analogies

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Plan

- Introduction to acoustics
  - Linearized conservation equations
  - Fundamental solutions of the homogeneous wave equation
  - Near- & far-field, acoustical compactness
  - Acoustical energy and impedance
  - Green’s function and integral solution

- Introduction to aeroacoustical analogies
  - Lighthill’s analogy, choice of the acoustical variable
  - Curle’s analogy and non-compact sources
  - Low-frequency numerical acoustics (in a nutshell)
  - Ffowcs Williams & Hawkings analogy, application to fan noise

- Vortex Sound Theory, choice of the source term
  - Powell’s analogy, Mohring’s analogy, conservative formulation
  - Application to vortex pairing described by wrong flow models
Introduction to acoustics

- Linearized conservation equations
- Fundamental solutions of the homogeneous wave equation
- Near- & far-field, acoustical compactness
- Acoustical energy and impedance
- Green’s function and integral solution
Continuity equation

For an infinitesimal fluid particle:

\[
\frac{1}{\rho} \frac{D\rho}{Dt} = -\nabla \cdot \mathbf{v}
\]
Material derivative

- Continuity equation in Lagrangian form:

\[
\frac{1}{\rho} \frac{D\rho}{Dt} = -\nabla \cdot \mathbf{v}
\]

- Material derivative:

\[
\frac{D\rho}{Dt} = \frac{\partial \rho}{\partial t} + (\mathbf{v} \cdot \nabla)\rho
\]

Eulerian form

\[
\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0
\]
Momentum equation

\[ \rho \frac{Dv}{Dt} = -\nabla \Pi + f \]

Acceleration of the fluid particle

External body forces applied to the fluid particle

External stresses applied to the fluid particle:

\[ \Pi_{ij} = \rho \delta_{ij} - \sigma_{ij} \]

Hydrostatic pressure

Viscous stresses

\[ \rho \frac{Dv}{Dt} = -\nabla p + \nabla \sigma + f \]
Linearization

- Continuity and momentum equations:

\[
\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = Q_m \\
\rho \frac{D\mathbf{v}}{Dt} = -\nabla p + \nabla \sigma + \mathbf{f}
\]

- Perturbations = deviations with respect to uniform and stagnant fluid:

\[
\rho = \rho_0 + \rho' \\
p = p_0 + p' \\
\mathbf{v} = \mathbf{v}_0 + \mathbf{v}' = \mathbf{v}'
\]

- At first order, the continuity and momentum equations become:

\[
\frac{\partial \rho'}{\partial t} + \rho_0 \nabla \cdot \mathbf{v}' = Q_m \\
\rho_0 \frac{\partial \mathbf{v}'}{\partial t} = -\nabla p' + \nabla \cdot \mathbf{\sigma}' + \mathbf{f}
\]
Acoustic sources

- Eliminate $\mathbf{v}'$ from the linearized conservation equations:

$$
\frac{\partial}{\partial t} \left\{ \frac{\partial \rho'}{\partial t} + \rho_0 \nabla \cdot \mathbf{v}' = Q_m \right\}
$$

$$
- \nabla \cdot \left\{ \rho_0 \frac{\partial \mathbf{v}'}{\partial t} = -\nabla p' + \nabla \cdot \mathbf{\sigma}' + \mathbf{f} \right\}
$$

$$
\frac{\partial^2 \rho'}{\partial t^2} - \nabla^2 p' = -\nabla \cdot \mathbf{f} - \nabla \cdot (\nabla \cdot \mathbf{\sigma}') + \frac{\partial Q_m}{\partial t}
$$

more unknowns than equations…
Linearized constitutive equation

- Equation of state: \( p = p(\rho, s) \)

- Perturbation: \( p' = \left( \frac{\partial p}{\partial \rho} \right)_s \rho' + \left( \frac{\partial p}{\partial s} \right)_\rho s' \)

- Definition of the speed of sound: \( c_0^2 \equiv \left( \frac{\partial p}{\partial \rho} \right)_s \)

\[ p' = c_0^2 \rho' + \left( \frac{\partial p}{\partial s} \right)_\rho s' \]
Sources of sound

$$\frac{1}{c_0^2} \frac{\partial^2 p'}{\partial t^2} - \nabla^2 p' = -\nabla \cdot f - \nabla \cdot (\nabla \cdot \sigma') + \frac{1}{c_0^2} \left( \frac{\partial p}{\partial s} \right) \rho \frac{\partial^2 s'}{\partial t^2} + \frac{\partial Q_m}{\partial t}$$

- Non-uniform force field
- Fluctuating viscous stresses
- Entropy fluctuations
- Fluctuating mass injection

Mass source is used as model for entropy production. We assume iso-kinetic injection (no momentum source) and isentropic process.
Homogeneous wave equation

- In absence of any source:
  - No external forces
  - Frictionless
  - Isentropic
  - Quiescent fluid

\[
\frac{1}{c_0^2} \frac{\partial^2 p'}{\partial t^2} - \nabla^2 p' = 0
\]

Acoustic field driven by initial and boundary conditions
Plane waves

- Propagation in $x_1$ direction: $p' = p'(x_1, t)$, $\mathbf{x} = (x_1, x_2, x_3)$

- Homogeneous wave propagation equation:

$$\frac{1}{c_0^2} \frac{\partial^2 p'}{\partial t^2} - \frac{\partial^2 p'}{\partial x_1^2} = 0$$

- General solution:

$$p' = f \left( t - \frac{x_1}{c_0} \right) + g \left( t + \frac{x_1}{c_0} \right)$$

Right-propagating wave

Left-propagating wave
Piston in a pipe

- Acoustic pressure induced by oscillating piston in semi-infinite pipe:

- Acoustical velocity given by linearized momentum equation:

\[ \rho_0 \frac{\partial v_1'}{\partial t} = -\frac{\partial p'}{\partial x_1} \]

- For right-propagating wave:

\[ p' = f \left( t - \frac{x_1}{c_0} \right) \quad v_1' = q \left( t - \frac{x_1}{c_0} \right) \quad q = \frac{1}{\rho_0 c_0} f \left( t - \frac{x_1}{c_0} \right) \]
Specific impedance

- General solution for acoustical pressure and velocity:

\[ p' = f \left( t - \frac{x_1}{c_0} \right) + g \left( t + \frac{x_1}{c_0} \right) \]

\[ v'_1 = \frac{1}{\rho_0 c_0} f \left( t - \frac{x_1}{c_0} \right) - \frac{1}{\rho_0 c_0} g \left( t + \frac{x_1}{c_0} \right) \]
Spherical waves

- Wave propagation equation in spherical coordinates:

\[
\frac{1}{c_0^2} \frac{\partial^2 (rp')}{\partial t^2} - \frac{\partial^2 (rp')}{\partial x_1^2} = 0
\]

- Same solution as in 1D using \( rp' \) as unknown:

\[
rp' = f \left( t - \frac{x_1}{c_0} \right) + g \left( t + \frac{x_1}{c_0} \right)
\]

  Outgoing wave  

  Incoming wave

- In frequency domain:

\[
p' = \frac{A}{r} \exp \left( i\omega \left( t - \frac{r}{c_0} \right) \right) = \frac{A}{r} \exp \left( i (\omega t - kr) \right)
\]
Far-field and near-field

- Linearized momentum equation in spherical coordinates:

\[
\rho_0 \frac{\partial v'_r}{\partial t} = - \frac{\partial p'}{\partial r} \quad \Rightarrow \quad v'_r = \frac{p'}{\rho_0 c_0} \left( 1 + \frac{1}{ikr} \right) \quad k = \omega / c_0
\]

- Two regimes:
  - Far-field:
    \[
    \lim_{kr \to \infty} v'_r = \frac{p'}{\rho_0 c_0} \propto \frac{1}{r}
    \]
    Plane wave behaviour
  - Near-field:
    \[
    \lim_{kr \to 0} v'_r = \frac{p'}{i\omega \rho_0 r} \propto \frac{1}{r^2}
    \]
    Locally incompressible flow
Acoustical compactness

- Upon normalization using the length scale $L$ and the time scale $\tau$: 
  \[ \tilde{t} \equiv t/\tau \quad \tilde{x}_i \equiv x_i/L \]

the wave propagation equation \[ \frac{\partial^2 \varphi'}{\partial \tilde{t}^2} - c_0^2 \frac{\partial^2 \varphi'}{\partial \tilde{x}_i^2} = 0 \]

becomes:
\[ \frac{\partial^2 \varphi'}{\partial \tilde{x}_i^2} = \left( \frac{L}{c_0\tau} \right)^2 \frac{\partial^2 \varphi'}{\partial \tilde{t}^2} = He^2 \frac{\partial^2 \varphi'}{\partial \tilde{t}^2} \]

with Helmholtz number \[ He \equiv \frac{L}{c_0\tau} = \frac{\omega L}{c_0} = \frac{2\pi L}{\lambda} = kL \]

- Compact region: \[ He \ll 1 \Rightarrow \nabla^2 \varphi' = 0 \]
  Laplace equation

---

At low Helmholtz numbers, i.e. in a compact region, the wave propagation equation reduces to the Laplace equation, describing an incompressible potential flow.

Corollary: an incompressible potential flow model solves the “acoustics problem” in a compact region.
Monopoles, dipoles, quadrupoles

- Monopole = pulsating sphere, jumping in a boat
  - Physically: unsteady combustion, pipe exhaust, vocal folds, ...

- Dipole = oscillating sphere, playing with a ball in a boat
  - Less efficient than monopole
  - Physically: unsteady forces

- Quadrupole = deforming sphere without change of volume nor net force, fighting in a boat!
  - Less efficient than dipole
  - Physically: turbulence
Acoustical energy

- Manipulating linearized conservation equations:

\[
\frac{p'}{\rho_0} \left\{ \frac{\partial \rho'}{\partial t} + \rho_0 \nabla \cdot \mathbf{v}' = Q_m \right\} \\
+ \mathbf{v}' \cdot \left\{ \rho_0 \frac{\partial \mathbf{v}'}{\partial t} = -\nabla p' + \nabla \cdot \sigma' + \mathbf{f} \right\}
\]

\[
\frac{\partial}{\partial t} \left( \frac{1}{2} \rho_0 (\mathbf{v}')^2 + \frac{1}{2} \frac{p'^2}{\rho_0 c_0^2} \right) + \nabla \cdot (p' \mathbf{v}') = \mathbf{v}' \cdot \mathbf{f} + \frac{p'}{\rho_0 c_0^2} \left( \frac{\partial p}{\partial s} \right)_\rho \frac{\partial s'}{\partial t} + \frac{p' Q_m}{\rho_0}
\]
Acoustical energy and intensity

\[
\frac{\partial E}{\partial t} + \nabla \cdot \textbf{I} = \textbf{v}' \cdot \textbf{f} + \frac{p'}{\rho_0 c_0^2} \left( \frac{\partial p}{\partial s} \right)_\rho \frac{\partial s'}{\partial t} + \frac{p'Q_m}{\rho_0}
\]

Forces (e.g. vibrating walls)

Volume source

Entropic processes (e.g. combustion)

\[
E = \frac{1}{2} \rho_0 (\textbf{v}')^2 + \frac{1}{2} \frac{p'^2}{\rho_0 c_0^2}
\]

Acoustic energy

\[
\textbf{I} = p' \textbf{v}'
\]

Acoustic intensity
CAUTION !!!

- We deduced an equation for quadratic functions of perturbation (E and I) from linear approximation !!!

- These definitions of energy and intensity are only valid in a uniform and stagnant fluid !!!
Integral formulation for steady harmonic oscillations

\[ \langle P \rangle = \iiint_S \langle I \cdot n \rangle \, dS = \iiint_V \langle \mathbf{v}' \cdot \mathbf{f} + \frac{p'Q_m}{\rho_0} \rangle \, dV \]

Power, averaged over one cycle

\[ \frac{1}{T} \int p' \, dV = \frac{1}{T} \int_0^T p' \, \frac{dV}{dt} \, dt \]

\[ \frac{1}{T} \int F \cdot \, dx = \frac{1}{T} \int_0^T F \cdot \frac{dx}{dt} \, dt \]
Effect of hard wall

\[ kh \ll 1 \quad \Rightarrow \quad p' = p_d' + p_r' \approx 2p_d' \]

\[ |I| \approx 4 \frac{(p_d')^2}{\rho_0 c_0} \]

\[ \langle P \rangle \approx \frac{1}{2} \left( 4\pi r^2 |I| \right) = 2 \left( 4\pi r^2 \frac{(p_d')^2}{\rho_0 c_0} \right) = 2 \langle P_0 \rangle \]
Source impedance

- The wall increases the pressure at the source:

\[ \langle P \rangle = \frac{1}{T} \int_0^T p' \, dV \]

\[ p' \approx 2 \, p'_d \]

The sound radiation is determined by the source and the impedance which it experiences!
Free field Green’s function in 3 dimensions

- Inhomogeneous wave equation:
  \[
  \frac{\partial^2 G}{\partial t^2} - c_0^2 \nabla^2 G = \delta(x - y) \delta(t - \tau)
  \]

- Solution:
  \[
  G(x, t|y, \tau) = \frac{\delta\left(t - \tau - \frac{|x - y|}{c_0}\right)}{4\pi c_0^2 |x - y|}
  \]

- Retarded (emission) time:
  \[
  \tau^* = t - \frac{|x - y|}{c_0}
  \]

- Important properties:
  - Dirac function $\rightarrow$ convenient to obtain an integral solution
  - Reciprocity: $G(x, t|y, \tau) = G(y, -\tau|x, -t)$
Other Green’s functions

- In a few cases: analytical Green’s functions
  - Infinite planes: image sources (semi-anechoic environment)
  - Semi-infinite plane (trailing edge noise)
  - Infinite straight ducts: rectangular, cylindrical, annular

- In other cases: semi-analytical Green’s functions
  - Compact (low-frequency) Green’s functions (Howe)
  - Wiener-Hopf technique, Schwarzchild’s technique (TE-LE backscattering, Roger)
  - Slowly-varying duct (Rienstra)

- In all other cases: numerical Green’s functions
  - Low-frequency techniques
    - Finite Element Methods, Boundary Element Methods
  - High-frequency techniques
    - Ray-tracing methods, Statistical Energy Analysis
  - Mid-frequency techniques
    - Multigrid techniques, fast multipole BEM, ...
Solution of the wave equation based on Green’s function

\[
\left\{ \frac{\partial^2 \rho'}{\partial t^2} - c_0^2 \nabla^2 \rho' = q(x, t) \right\} \times G, \iiint_V, \int_{t_0}^t
\]

\[
- \left\{ \frac{\partial^2 G'}{\partial t^2} - c_0^2 \nabla^2 G = \delta(x - y) \delta(t - \tau) \right\} \times \rho', \iiint_V, \int_{t_0}^t
\]

\[
\rho'(x, t) = \int_{t_0}^t \iiint_V q(y, \tau) G(x, t|y, \tau) \, d^3yd\tau
\]

\[
+ \int_{t_0}^t \iiint_V \left( \rho'(y, \tau) \frac{\partial^2 G}{\partial \tau^2} - G \frac{\partial^2 \rho'(y, \tau)}{\partial \tau^2} \right) \, d^3yd\tau
\]

\[
- c_0^2 \int_{t_0}^t \iiint_V \left( \rho'(y, \tau) \frac{\partial^2 G}{\partial y_i^2} - G \frac{\partial^2 \rho'(y, \tau)}{\partial y_i^2} \right) \, d^3yd\tau
\]
Integral solution of the wave equation

- Integrating by parts:
  \[ \rho'(x, t) = \int_{t_0}^{t} \int \int \int_V q(y, \tau) \ G(x, t | y, \tau) \ dy \ d\tau \]

  \[ - \left[ \int \int \int_V \left( \rho'(y, \tau) \ \frac{\partial G}{\partial \tau} - G \ \frac{\partial \rho'(y, \tau)}{\partial \tau} \right) \ dy \right]_{\tau=t_0} \]

  \[ - c_0^2 \int_{t_0}^{t} \int \int_{S} \left( \rho'(y, \tau) \ \frac{\partial G}{\partial y_i} - G \ \frac{\partial \rho'(y, \tau)}{\partial y_i} \right) \ n_i \ dy \ d\tau \]

- Further simplifications
  - Silent initial conditions
  - Tailored Green’s function

\[ \rho'(x, t) = \int_{t_0}^{t} \int \int \int_V q(y, \tau) \ G(x, t | y, \tau) \ dy \ d\tau \]

Having an integral formulation improves the numerical stability of the prediction when detailed flow data (e.g. LES) are available, and otherwise permits deriving scaling laws!
Summary

- Assuming small amplitude acoustic perturbations, the equations of fluid motion can be linearized and used to derive a wave equation for these perturbations.

- The relationship between the perturbations are given by the linearized momentum equation and the linearized constitutive equation:

  \[ \rho_0 \frac{\partial \nabla'}{\partial t} = -\nabla p' \]

  \[ p' = c_0^2 \rho' + \left( \frac{\partial p}{\partial s} \right)_\rho s' \]

- In the linear approximation, the sources of the acoustic field can be due to:
  - Unsteady mass injection or entropy fluctuations → monopolar character.
  - Non-uniform forces → dipolar character.
  - Fluctuating Reynolds stresses → quadrupolar character.

- Each of these sources has a different radiation efficiency in free field.

- The sound radiation is determined by the source and the impedance which it experiences!

- An integral formulation of the wave equation can be obtained using Green’s functions, which enhances the numerical robustness of the prediction.
Aeroacoustic analogies

- Lighthill’s analogy
- Curle’s analogy
- Ffowcs Williams & Hawkings analogy
Why not Direct CAA?

- Acoustic field = part of the flow field ➔ most straightforward approach: Computational AeroAcoustics (CAA)
  - But: at low Mach numbers: orders of magnitude of difference between
    - Length scales: $\lambda_{ac} = L_{turb} / M$
    - Magnitudes: $O(M^4)$ of the flow energy radiates into the far field
  - High order schemes needed to capture acoustic propagation
  - Numerical cost of a direct CAA scales with $Re^2 M^{-4}$ for a Large Eddy Simulation
  - Specific issues related to CFD discretisation techniques applied to acoustics
    - Dissipation and dispersion errors
    - Initial and boundary conditions
Lighthill’s aeroacoustical analogy: concept

- The problem of sound produced by a turbulent flow is, **from the listener’s point of view**, analogous to a problem of propagation in a uniform medium at rest in which equivalent sources are placed.

- Wave propagation region: linear wave operator applies

\[ \frac{\partial^2 \rho'}{\partial t^2} - c_0^2 \frac{\partial^2 \rho'}{\partial x_i^2} = 0 \]

- Turbulent region: fluid mechanics equations apply

\[
\begin{align*}
\frac{\partial \rho}{\partial t} + \frac{\partial \rho v_i}{\partial x_i} &= 0 \\
\frac{\partial \rho v_i}{\partial t} + \frac{\partial \rho v_i v_j}{\partial x_j} &= -\frac{\partial \Pi_{ij}}{\partial x_j} \\
\Pi_{ij} &= p \delta_{ij} - \sigma_{ij}
\end{align*}
\]
Lighthill’s analogy: formal derivation

\[ \frac{\partial}{\partial t} \left\{ \frac{\partial \rho}{\partial t} + \frac{\partial \rho v_i}{\partial x_i} \right\} = 0 \]

\[ -\frac{\partial}{\partial x_i} \left\{ \frac{\partial \rho v_i}{\partial t} + \frac{\partial \rho v_i v_j}{\partial x_j} \right\} = -\frac{\partial p}{\partial x_i} + \frac{\partial \sigma_{ij}}{\partial x_j} \]

\[ \frac{\partial^2 \rho}{\partial t^2} = \frac{\partial^2 (\rho v_i v_j - \sigma_{ij})}{\partial x_i \partial x_j} + \frac{\partial^2 p}{\partial x_i^2} \]

\[ \frac{\partial^2 \rho}{\partial t^2} - c_0^2 \frac{\partial^2 \rho}{\partial x_i^2} = \frac{\partial^2 (\rho v_i v_j - \sigma_{ij})}{\partial x_i \partial x_j} + \frac{\partial^2 (p - c_0^2 \rho)}{\partial x_i^2} \]
Lighthill’s aeroacoustical analogy: reference state

- Reformulation of fluid mechanics equations, and use of arbitrary speed $c_0$:

$$\frac{\partial^2 \rho}{\partial t^2} - c_0^2 \frac{\partial^2 \rho}{\partial x_i^2} = \frac{\partial^2 (\rho v_i v_j - \sigma_{ij})}{\partial x_i \partial x_j} + \frac{\partial^2 (p - c_0^2 \rho)}{\partial x_i^2}$$

- Definition of a reference state:

\[
\begin{align*}
\rho' & \equiv \rho - \rho_0 \\
p' & \equiv p - p_0 \\
v'_i & \equiv v_i
\end{align*}
\]

- Aeroacoustical analogy:

$$\frac{\partial^2 \rho'}{\partial t^2} - c_0^2 \frac{\partial^2 \rho'}{\partial x_i^2} = \frac{\partial^2 T_{ij}}{\partial x_i \partial x_j}$$

with \( T_{ij} = \rho v_i v_j + (p' - c_0^2 \rho') \delta_{ij} - \sigma_{ij} \)

Lighthill’s tensor

*Exact... and perfectly useless!*
Sound produced by free isothermal turbulent flows at low Mach number

- Solution using Green’s fct

\[ \rho'(x, t) = \int_{-\infty}^{t} \int \int \int_{V} \frac{\partial^2 T_{ij}}{\partial y_i \partial y_j} G \, d^3y \, d\tau - c_0^2 \int_{-\infty}^{t} \int \int_{\partial V} \left( \frac{\partial T_{ij}}{\partial y_i} - G \frac{\partial \rho'}{\partial y_i} \right) n_i \, d^2y \, d\tau \]

- Purpose: simplify the RHS

\[ \frac{\partial^2 T_{ij}}{\partial x_i \partial x_j} \]

- High Reynolds number
- Isentropic
- Low Mach number

\[ \rho v_i v_j \simeq \rho_0 v_i v_j \]

- Using free field Green’s fct

\[ G_0(t, x|\tau, y) = \frac{\delta(t - \tau - |x - y|/c_0)}{4\pi c_0^2 |x - y|} \]

\[ \rho'(x, t) = \frac{\partial^2}{\partial x_i \partial x_j} \int \int \int_{V} \left[ \frac{\rho_0 v_i v_j}{4\pi c_0^2 |x - y|} \right] d^3y \]

Quadrupolar source

\[ t^* = t - |x - y|/c_0 \]
Lighthill’s $M^8$ law

- **Integral solution:**
  \[
  \rho'(x, t) = \frac{\partial^2}{\partial x_i \partial x_j} \int \int \int_V \left[ \frac{\rho_0 v_i v_j}{4\pi c_0^2 |x - y|} \right] d^3y
  \]

- **Scaling law:**
  \[
  t^* = t - \frac{|x - y|}{c_0}
  \]

- **Acoustic scale:**
  \[x \propto \lambda = \frac{f}{c_0}\]

- **Flow time scale:**
  \[D / U_0\]

- **Spatial derivative:**
  \[U_0 / (c_0 D)\]

- **Acoustical power:**
  \[
  W = \frac{4\pi |x|^2 p'^2}{\rho_0 c_0^2} \propto \rho_0 c_0^3 D^2 M^8
  \]
Choice of the aeroacoustical variable

- Manipulating the mass and momentum equations yields:

\[
\frac{\partial^2 \rho}{\partial t^2} = \frac{\partial^2}{\partial x_i \partial x_j} (\rho v_i v_j - \sigma_{ij}) + \frac{\partial^2 p}{\partial x_i^2}
\]

- From there, two choices are possible for the acoustical variable:
  - Acoustical density perturbation:
    \[
    \frac{\partial^2 \rho'}{\partial t^2} - c_0^2 \frac{\partial^2 \rho'}{\partial x_i^2} = \frac{\partial^2}{\partial x_i \partial x_j} (\rho v_i v_j - \sigma_{ij}) + \frac{\partial^2}{\partial x_i^2} (p' - c_0^2 \rho')
    \]

  - Isentropic noise generation

  - Acoustical pressure perturbation:
    \[
    \frac{1}{c_0^2} \frac{\partial^2 p'}{\partial t^2} - \frac{\partial^2 p'}{\partial x_i^2} = \frac{\partial^2}{\partial x_i \partial x_j} (\rho v_i v_j - \sigma_{ij}) + \frac{\partial^2}{\partial t^2} \left( \frac{p'}{c_0^2} - \rho' \right)
    \]

  - Combustion noise
Curle’s analogy: fixed rigid bodies

- Lighthill’s aeroacoustical analogy:
  \[ \frac{\partial^2 \rho'}{\partial t^2} - c_0^2 \frac{\partial^2 \rho'}{\partial x_i^2} = \frac{\partial^2 T_{ij}}{\partial x_i \partial x_j} \]

- Integral solution using Green’s function
  \[ \rho'(x, t) = \int_{-\infty}^{t} \int \int_{V} \frac{\partial^2 T_{ij}}{\partial y_i \partial y_j} G \, d^3 y \, d\tau \]  
  - incident field
  \[ - c_0^2 \int_{-\infty}^{t} \int \int_{\partial V} \left( \rho' \frac{\partial G}{\partial y_i} - G \frac{\partial \rho'}{\partial y_i} \right) n_i \, d^2 y \, d\tau \]  
  - scattered field

- Partial integration of source integral
  \[ \int_{-\infty}^{t} \int \int_{V} \frac{\partial^2 T_{ij}}{\partial y_i \partial y_j} G \, d^3 y \, d\tau = \int_{-\infty}^{t} \int \int_{V} T_{ij} \frac{\partial^2 G}{\partial y_i \partial y_j} \, d^3 y \, d\tau \]

- Curle’s analogy: uses free field Green’s function
  \[ G_0(t, x | \tau, y) = \frac{\delta(t - \tau - |x - y|/c_0)}{4\pi c_0^2 |x - y|} \]

  \[ \rho'(x, t) = \frac{\partial^2}{\partial x_i \partial x_j} \int \int_{V} \left[ \frac{T_{ij}}{4\pi c_0^2 |x - y|} \right] \, d^3 y - \frac{\partial}{\partial x_i} \int \int_{\partial V} \left[ \frac{p'n_i}{4\pi c_0^2 |x - y|} \right] \, d^2 y \]
A popular formulation for industrial applications

- Curle’s formulation is quite powerful
  - It enforces the correct radiation pattern of each source component:
    $$ P_{\text{quadru}} / P_{\text{dipo}} \sim M^2 $$
  - At low Mach numbers, dipolar contribution dominates the quadrupolar one for compact sources
  - Surface scalar ($\rho'$) data are much less demanding in memory than volumetric, tensorial ($T_{ij}$) data
  - Surface mesh often available from design stage

$$ 4\pi c_0^2 \rho'(\mathbf{x}, t) = \frac{\partial^2}{\partial x_i \partial x_j} \int \int \int_{V} \left[ \frac{T_{ij}}{|\mathbf{x} - \mathbf{y}|} \right] \, d^3y + \frac{\partial}{\partial x_i} \int \int_{\partial V} \left[ \frac{p' n_i}{|\mathbf{x} - \mathbf{y}|} \right] \, d^2y $$

\begin{align*}
\text{Quadrupole, } W \propto M^8 \\
\text{in free field}
\end{align*}

\begin{align*}
\text{Dipole, } W \propto M^6 \\
\text{in free field}
\end{align*}

- BUT: tricky implementation for non-compact geometries...
The hybrid approach from a practical viewpoint

- The computation of flow is decoupled from the computation of sound
  - Acoustic prediction: post-processing of source field data
- Fundamental assumption: one-way coupling
  - Unsteady flow produces sound and affects its propagation
  - BUT: sound waves do not affect flow field significantly
  - Principal application of the hybrid approach: flows at low Mach numbers
- Usable simulation tools for the flow description:
  - Reynolds Averaged Navier-Stokes (RANS) solver
    → time-averaged data (SNGR to reconstruct transient)
  - Unsteady RANS → unsteady, but only large scale
  - Large Eddy Simulation (LES), Detached Eddy Simulation (DES)
    → unsteady, broadband turbulence (up to grid & scheme cut-off frequency)
- Low-Mach number applications
  - Incompressible LES / DES solvers to reduce CPU cost
  - Careful interpretation of the flow data in aeroacoustical analogy
What can go wrong?

Geometry & flow conditions

Transient CFD simulation

Hydrodynamic field

Aeroacoustical analogy

Equivalent sources

Numerical acoustics

Acoustic far-field
Mapping: coarsening and direction errors

Under-sampling $\rightarrow$ information loss

Element normal changes $\rightarrow$ directivity
Conservative vs non-conservative mapping schemes

**Conservation principle:**
*same integrated source term on both meshes*

**Non-conservative scheme**
- Nearest-nodes scheme, loop on the coarse mesh

→ information loss

---

**Conservative scheme**
- Nearest-nodes scheme, loop on the fine mesh

- Distance-based scheme
Error depends strongly on sampling!

Error is much lower, and does not depend on sampling alone, but on acoustic effects.

\[
\text{error} = \frac{|p_{ac} - p_{ac}^{\text{ref}}|}{|p_{ac}^{\text{ref}}|}
\]
Truncation error

- Numerical effort focused on “active” regions of the source field
- Lighthill’s analogy $\rightarrow$ extended quadrupolar field
- Wakes convected at subsonic speeds are not efficient sources of sound $\rightarrow$ truncation to reduce effort
- BUT: induces spurious source terms!
- Two approaches to mitigate issue:
  - Correction term
  - Windowing
Curle’s analogy applied to compact sources

- Sound generated by obstacle in flow
  - Side mirror
  - Antenna
  - Sunroof spoiler
  - HVAC vent grids
  - Landing gear

- Acoustical compactness expressed by Helmholtz number: $He = \frac{2\pi f D}{c_0}$

- Flow unsteadiness expressed by Strouhal number: $Sr = \frac{f D}{U} = O(1)$

- Acoustical compactness depends on Mach number: $He = \frac{2\pi Sr U}{c_0} = 2\pi Sr M$

- $He \ll 1$:
  - Compact body, does not scatter its own sound field
  - Neglecting scattering integral is not a significant error
While, for non-compact sources (ducted flows...)

[Diagram of ducted flows and circular pattern]
Example: HVAC duct

- Acoustic mesh generation
- CFD pressure fluctuations
  - Imported in time domain (binary format)
  - Fourier-transformed
  - Mapped onto acoustic mesh

Acoustical half-wavelength

'More' acoustics is captured by the low-order transient CFD
Low-frequency numerical acoustics

- Finite Element Method
- Boundary Element Method
- BEM/Curle formulation
Low-frequency numerical acoustics in a nutshell

- Solution of generic inhomogeneous wave propagation equation

\[
\nabla^2 p' - \frac{1}{c_0^2} \frac{\partial p'}{\partial t^2} = -\rho_0 \frac{\partial q'}{\partial t}
\]

using Fourier decomposition for source and unknown:

- Helmholtz equation

\[
\nabla^2 \hat{p} + k^2 \hat{p} = -i\omega \rho_0 \hat{q}
\]

with appropriate boundary conditions:

\[
\begin{align*}
p(r) &= \overline{p}(r) \quad &r \in \Omega_p \\
v_n(r) &= \frac{i}{\rho_0 \omega} \frac{\partial p(r)}{\partial n} = \overline{v}_n(r) \quad &r \in \Omega_v \\
p(r) &= \overline{Z}(r) v_n(r) \quad &r \in \Omega_Z
\end{align*}
\]
Finite Element Method (FEM)

- Common numerical technique for solving many engineering problems
- Find a discrete distribution of field variables in a continuum domain, governed by partial differential equations and boundary conditions $\rightarrow$ set of algebraic equations
- Time-harmonic acoustic problems: nodal approximation of the pressure field

$$ p(x, y, z) = \sum_{j=1}^{n} N_j(x, y, z) p_j = [N_j] \{p_j\} $$

- Resolution of the Helmholtz equation via weighted residual method (weak form)
  $\rightarrow$ algebraic system of equations

$$ ( [K_a] + i\omega [C_a] - \omega^2 [M_a] ) \{p_j\} = \{F_a\} $$

- Convergence: completeness and compatibility $\rightarrow$ conforming elements
- Popular shape functions
  - Linear & quadratic tria’s and quad’s in 2D
  - Linear & quadratic tetra’s and hexa’s in 3D
FEM properties: pros and cons

- **Pros:**
  - Local shape functions $\rightarrow$ sparse matrices, can be arranged in band structure using ad-hoc numbering of the nodes
  - Matrices are symmetrical $\rightarrow$ good for memory and CPU usage
  - Shape functions are independent of frequency
    - mass and stiffness matrix also,
    - though damping matrix depends usually on frequency through impedance BCs
  - Real shape functions
    - mass and stiffness matrix also,
    - damping matrix is often complex through impedance BCs
  - FEM can handle inhomogeneous acoustic propagation medium
  - Numerical evaluation of the matrices is quite straightforward

- **Cons:**
  - Low-order shape functions $\rightarrow$ large number of elements to represent acoustic field
    - Rule of thumb: between 6 and 10 elements per wavelength
    - Numerical cost increases with both model size and frequency
  - Smaller accuracy of secondary variables (acoustical velocity)
  - Finite element size $\rightarrow$ in principle restricted to bounded domains
Boundary Element Method (BEM)

- Same discretisation principle as for FEM, but
  - Field variables inside the domain are related to their values on the domain boundary
- Direct and Indirect Boundary Element Method

**Direct Boundary Element Method (DBEM)**

- Derivation essentially similar to Curle’s analogy
  - Resolution of the Helmholtz equation \( \nabla^2 \hat{p}_a + k^2 \hat{p}_a = q_L \)
    
    with the source term \( \hat{q}_L = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left( - \frac{\partial^2 T_{ij}}{\partial x_i \partial x_j} \right) e^{-i\omega t} \, dt \equiv - \frac{\partial^2 T_{ij}}{\partial x_i \partial x_j} \)

    using free field Green’s function \( G = \frac{e^{-ikr}}{4\pi r} \), \( r = |x - y| \)
  
  \[ \nabla^2 G + k^2 G = -\delta(x - y) \]

- Collocation method: the integral solution is evaluated over the acoustic mesh
  - Exclude Green’s kernel singularity!

\[
\int_{\Omega_{\text{ac}}} \left( \nabla^2 p_a \right) \left( G - p_a \nabla^2 G \right) \, d^3 y = \int_{\Omega_{\text{ac}}} q_L \, G \, d^3 y + \int_{\Omega_{\text{ac}}} p_a \, \delta(x - y) \, d^3 y
\]

\[
C(x)p_a(x) = \int_{\partial\Omega} \left( \frac{\partial p_a}{\partial n} G - p_a \frac{\partial G}{\partial n} \right) \, d^2 y + \lim_{\epsilon \to 0} \int_{\Omega_{\text{ac}}} \frac{\partial^2 T_{ij}}{\partial y_i \partial y_j} \, G \, d^3 y
\]
**BEM vs. FEM**

- **BEM advantages:**
  - Naturally suited for exterior propagation problems (Sommerfeld radiation condition)
  - Same accuracy for acoustical pressure and acoustical velocity
  - Only the surface must be meshed
    - Smaller model size
    - Less human-intensive

- **FEM advantages:**
  - BEM matrices are
    - fully populated (IBEM: symmetrical)
    - Complex (complex Green’s function)
    - Frequency-dependent: must be computed for each frequency (however, this can be vastly improved using Fast Multipole Multigrid approaches)
  - Non-uniqueness of the solution for exterior problems with closed boundary surfaces
    - Nowadays resolved using over-determination nodes or Burton-Miller technique
  - Attention must be paid to the evaluation of the singularities in BEM

- **Conclusion: it depends...**
  - Exterior problems: BEM
  - Interior problems: FEM, excepted when...
Curle’s analogy with listener in the source field

\[ C(x) p_a(x) = \iiint_{\partial V} \left( \frac{\partial p_a}{\partial n} G - p_a \frac{\partial G}{\partial n} \right) \, d^2y + \lim_{\varepsilon \to 0} \iiint_{V \setminus V_\varepsilon} \frac{\partial^2 T_{ij}}{\partial y_i \partial y_j} G \, d^3y \]

- Integration by parts of the volume integral

\[ C(x) p_a(x) = \iiint_V T_{ij} \frac{\partial^2 G}{\partial y_i \partial y_j} \, d^3y + \iiint_{\partial V} \left( -c_0^2 \frac{\partial \rho_L}{\partial n} G - (p_L - c_0^2 \rho_L) \frac{\partial G}{\partial n} \right) \]
\[ - C(x) (p_L - c_0^2 \rho_L) + \iiint_{\partial V} \left( \frac{\partial p_a}{\partial n} G - p_a \frac{\partial G}{\partial n} \right) \, d^2y \]

- And since \( c_0^2 \rho_L = p_a \), we find

\[ C(x) p_L(x) = \iiint_V T_{ij} \frac{\partial^2 G}{\partial y_i \partial y_j} \, d^3y - \iiint_{\partial V} p_L \frac{\partial G}{\partial n} \, d^2y \]

When the listener is located in the source region, we obtain an integral solution for the pressure, irrespectively of its acoustic or hydrodynamic nature.

Curle’s classical solution for the acoustic pressure is recovered by placing the listener in the quiescent propagation region.
Contributions of compact and non-compact regions

- We decompose the pressure into acoustic and hydrodynamic: \( p_L = p_h + p_a \)

\[
C(\mathbf{x}) \ (p_h(\mathbf{x}) + p_a(\mathbf{x})) = \iiint_{V_1} T_{ij} \frac{\partial^2 G}{\partial y_i \partial y_j} \, d^3y - \iiint_{\partial V_1} (p_h + p_a) \frac{\partial G}{\partial n} \, d^2y \\
+ \iiint_{V_2} T_{ij} \frac{\partial^2 G}{\partial y_i \partial y_j} \, d^3y - \iiint_{\partial V_2} (p_h + p_a) \frac{\partial G}{\partial n} \, d^2y \tag{1}
\]

- Over compact region \( V_1 \):

\[
C(\mathbf{x}) p_h(\mathbf{x}) = \iiint_{V_1} T_{ij} \frac{\partial^2 G}{\partial y_i \partial y_j} \, d^3y - \iiint_{\partial V_1} p_h \frac{\partial G}{\partial n} \, d^2y \tag{2}
\]

- Subtracting (2) from (1):

\[
C(\mathbf{x}) p_a(\mathbf{x}) = -\iiint_{\partial V} p_a \frac{\partial G}{\partial n} \, d^2y \\
+ \iiint_{V_2} T_{ij} \frac{\partial^2 G}{\partial y_i \partial y_j} \, d^3y - \iiint_{\partial V_2} p_h \frac{\partial G}{\partial n} \, d^2y
\]
Validation: spinning vortex pair in straight duct

- **Purpose**: perform unambiguous validation of the BEM / Curle approach
  - Flow should be amenable to (nearly) exact modelling
  - Geometry should allow an exact evaluation of the scattering (tailored Green’s fct)
  - Incompressible flow model

- Leapfrogging of 2 rectilinear vortex filaments in an infinite 2D duct
  - Flow kinematics: based on the complex potentials of the system of 2 vortices, and of the infinite series of image vortices
  - Non-penetration, slip condition at both walls (streamlines)

- **Flow parameters**:
  - Duct height: $h = 1$ m
  - Initial position of the vortices: on centreline, distant by $d = h / 2$
  - Circulation: $\Gamma = 85$ m$^2$ s$^{-1}$
  - Mach number: $Ma = \Gamma / (d \ c_0) = 0.5$
Flow model: source fields and vortex desingularization

- Volume source field:
  1. Compute the vortex kinematics by time marching the equations

\[
\begin{align*}
  u_m &= -\frac{\Gamma}{4h} \left\{ \frac{\sin \left[ \pi \left( y_m - y_n \right) / h \right]}{\cosh \left[ \pi \left( x_m - x_n \right) / h \right] - \cos \left[ \pi \left( y_m - y_n \right) / h \right]} \\
  &\quad + \frac{\sin \left[ \pi \left( y_m + y_n \right) / h \right]}{\cosh \left[ \pi \left( x_m - x_n \right) / h \right] + \cos \left[ \pi \left( y_m + y_n \right) / h \right]} \\
  &\quad + \frac{\sin \left( 2\pi y_m / h \right)}{1 + \cos \left( 2\pi y_m / h \right)} \right\} \\
  v_m &= -\frac{\Gamma}{4h} \left\{ \frac{-\sinh \left[ \pi \left( x_m - x_n \right) / h \right]}{\cosh \left[ \pi \left( x_m - x_n \right) / h \right] - \cos \left[ \pi \left( y_m - y_n \right) / h \right]} \\
  &\quad + \frac{\sinh \left[ \pi \left( x_m - x_n \right) / h \right]}{\cosh \left[ \pi \left( x_m - x_n \right) / h \right] + \cos \left[ \pi \left( y_m + y_n \right) / h \right]} \right\}
\end{align*}
\]

2. Compute the induced velocity field using the desingularized kernel

\[
v_\theta (r) = \frac{\Gamma}{2\pi r} \left[ 1 - \exp \left( -\frac{r^2}{2\sigma^2} \right) \right]
\]

- Wall pressure: integrated from unsteady Bernoulli’s eq.: 

\[
p_w = -\rho \left( \frac{\partial \Phi_w}{\partial t} + \frac{w_w^2}{2} \right)
\]
Source fields: $T_{xy}$ and wall pressure
Fourier content of the source field

- Fourier transform of Lighthill’s tensor $T_{ij} \approx \rho_0 uv$ at the position $(0.25,0)$

- Control parameters:
  - Circulation $I \rightarrow$ frequency shift of entire spectrum, and Mach number
  - Initial spacing $d \rightarrow$ modulation due to the image vortices
    $\rightarrow$ frequency shift of entire spectrum, Mach number and acoustical compactness
Implementation in DBEM solver

- BEM mesh
  - \((x,y,z) = ([-5,5],[0,1],[-0.1,0.1]) \ h\)
  - Non-reflective boundary conditions at both ends: \(Z = \rho_0 c_0\)
  - Refined in source region for quadrupoles

- Quadrupoles integrated over cells with \((h / 20)\) dimensions in \((x,y)\) plane
- Dipoles generated over lower and upper wall surfaces
How much error due to the desingularization?

- Validation: test on a simple test case: two leapfrogging vortex filaments in unbounded domain
  - Quantitative validation with far field solution (Howe, 2003):
    \[
    p' \approx -4 \sqrt{\frac{\pi d}{2r}} \rho_0 U^2 M^{3/2} \cos \left[ 2 \theta - 2\Omega \left( t - \frac{r}{c_0} \right) + \frac{\pi}{4} \right]
    \]
- Variation of the core size over a fixed mesh with spatial sampling $\Delta$
- Two trends:
  - Small values of $\sigma/\Delta$: numerical noise increases due to lack of resolution
  - Large values of $\sigma/d$: sound prediction decreases due to smeared vortex interaction

\[\sigma/d = 0.1\]
Reference solution used for validation

- Reference solution: based on tailored Green’s function

\[
G_1 = \frac{i}{2h} \sum_{n=0}^{\infty} \frac{1}{C_n k_n} \cos(\eta_n y_0) \cos(\eta_n y) e^{\pm i k_n (x-x_0)}
\]

\[
C_n = \begin{cases} 
1 & \text{if } n = 0 \\
1/2 & \text{if } n \neq 0
\end{cases}
\]

\[
p_a(x, \omega) = -\frac{i}{2h} \sum_{n=0}^{\infty} \frac{\cos(\eta_n y)}{k_n C_n} \left\{ k_n^2 \iint_{S_0} \cos(\eta_n y_0) \rho_0 u^2 e^{\pm i k_n (x-x_0)} \, dx_0 \, dy_0 \\
+ \eta_n^2 \iint_{S_0} \cos(\eta_n y_0) \rho_0 v^2 e^{\pm i k_n (x-x_0)} \, dx_0 \, dy_0 \\
\pm i k_n \eta_n \iint_{S_0} \sin(\eta_n y_0) \rho_0 u v e^{\pm i k_n (x-x_0)} \, dx_0 \, dy_0 \right\}
\]
Validation of the Curle/DBEM method for $kh = 4.8$

Reference solution

Curle/DBEM

Dipolar contribution

Quadrupolar contribution

Lighthill
Validation of the Curle/DBEM method at \((x,y) = (3,0.5)\)

- Overall quite good agreement at all frequencies, excepted just above cut-off frequency, due to inaccurate boundary condition.
Ffowcs Williams and Hawkings analogy: moving bodies

- Use of generalized functions to account for body motion
- Body motion trajectory described by function \( F(x,y,z,t) \):

\[
\begin{align*}
F &= 0 \\
|\nabla F| &= 1 \\
F > 0 &
\end{align*}
\]

- Heaviside and Dirac functions properties:

\[
\begin{align*}
H(F) &= 0 & \text{inside solid body} \\
    &= 1 & \text{in fluid region} \\
\nabla (H(F)) &= \delta(F') \nabla F \\
\frac{\partial}{\partial t} (H(F)) &= \delta(F) \frac{\partial F}{\partial t}
\end{align*}
\]
Conservative equations and analogy using generalized functions

- Equations of conservation of mass and momentum can be rewritten as:

\[
\frac{\partial (\rho' H)}{\partial t} + \frac{\partial (\rho v_i H)}{\partial x_i} = \rho_0 (\mathbf{v} \cdot \nabla F) \delta(F) = \rho_0 V_n \delta(F)
\]

\[
\frac{\partial (\rho v_i H)}{\partial t} + \frac{\partial}{\partial x_j} \left[(\rho v_i v_j + p \delta_{ij} + \sigma_{ij}) H\right] = (p \delta_{ij} + \sigma_{ij}) \nabla F \delta(F)
\]

- The analogy becomes:

\[
\frac{\partial^2 (\rho' H)}{\partial t^2} - c_0^2 \frac{\partial^2 (\rho' H)}{\partial x_i^2} = \frac{\partial^2}{\partial x_i \partial x_j} (T_{ij} H) - \frac{\partial}{\partial x_i} \left[(p \delta_{ij} + \sigma_{ij}) \nabla F \delta(F)\right] + \frac{\partial}{\partial t} (\rho_0 V_n \delta(F))
\]
Integral solution

- Using the free-field Green’s function:

\[
\rho'(x, t) = \frac{\partial^2}{\partial x_i \partial x_j} \int_{-\infty}^{t} \int \int \int_V \frac{\delta(t - \tau - |x - y|/c_0)}{4\pi c_0^2 |x - y|} T_{ij}(y, \tau) \, d^3y \, d\tau
\]

\[
- \frac{\partial}{\partial x_i} \int_{-\infty}^{t} \int \int_{\partial V} \frac{\delta(t - \tau - |x - y|/c_0)}{4\pi c_0^2 |x - y|} [p \, \delta_{ij} + \sigma_{ij}] (y, \tau) n_j \, d^2y \, d\tau
\]

\[
+ \frac{\partial}{\partial t} \int_{-\infty}^{t} \int \int_{\partial V} \frac{\delta(t - \tau - |x - y|/c_0)}{4\pi c_0^2 |x - y|} \rho_0 V_n(y, \tau) \, d^2y \, d\tau
\]

- More practical to have the source described in a moving coordinate system \( \eta \) attached to the body.

- Retarded time equation:

\[
g(t^*, t, x, \eta) \equiv t - t^* - \frac{|x - y(\eta, t^*)|}{c_0} = 0
\]

- Dirac function property:

\[
\int_{-\infty}^{\infty} \delta(h(\xi)) \, f(\xi) \, d\xi = \sum_i \frac{f(\xi_i)}{|h'(\xi_i)|}, \quad h(\xi_i) = 0
\]
Doppler effects

- Applied to the quadrupolar component:
  \[
  \int^{t}_{-\infty} \int \int_{V_\eta} \frac{\delta(g(\tau, t, x, \eta))}{4\pi c_0^2|x - y(\eta, \tau)|} T_{ij} \, d^3\eta \, d\tau = \frac{1}{4\pi c_0^2} \int \int_{V_\eta} \left[ \frac{T_{ij}}{R|1 - \mathbf{M}.\mathbf{R}/R|} \right] \, d^3\eta
  \]
  \[\mathbf{M}: \text{vector Mach number} \quad 1 - \mathbf{M}.\mathbf{R}/R: \text{Doppler factor}\]

- Final solution:
  \[
  \rho'(x, t) = \frac{1}{4\pi c_0^2} \frac{\partial^2}{\partial x_i \partial x_j} \int \int_{V_\eta} \left[ \frac{T_{ij}}{R|1 - \mathbf{M}.\mathbf{R}/R|} \right] \, d^3\eta
  \]
  \[- \frac{1}{4\pi c_0^2} \frac{\partial}{\partial x_i} \int \int_{\partial V_\eta} \left[ \frac{(p \delta_{ij} + \sigma_{ij}) n_j}{R|1 - \mathbf{M}.\mathbf{R}/R|} \right] \, d^2\eta
  \]
  \[+ \frac{1}{4\pi c_0^2} \frac{\partial}{\partial t} \int \int_{\partial V_\eta} \left[ \frac{\rho_0 V_n}{R|1 - \mathbf{M}.\mathbf{R}/R|} \right] \, d^2\eta
  \]

- The flow intrinsic features (flow separation, turbulent transition, ...) are expressed in the frame of reference attached to the moving axes.

- The Doppler effects (convective amplification and frequency shift) are given by the motion of the sources in the fixed coordinate system.
Rotating point force

- FW-H analogy in time domain:

\[
\rho'(x, t) = \int_{-\infty}^{t} \int_{S(\tau)} \frac{\partial G}{\partial y_i} f_i \, d^2y \, d\tau + \int_{-\infty}^{t} \int_{V(\tau)} \frac{\partial^2 G}{\partial y_i \partial y_j} T_{ij} \, d^3y \, d\tau + \int_{-\infty}^{t} \int_{S(\tau)} \frac{\partial G}{\partial \tau} \rho_0 V_n \, d^2y \, d\tau
\]

- Using the free field Green’s function, and for a compact source:

\[
\rho'(x, t) \sim -\frac{1}{4\pi c_0^2} \frac{\partial}{\partial x_i} \int_{S(\tau_0)} \left[ \frac{f_i}{R|1 - \mathbf{M} \mathbf{R}/R|} \right]_{\tau^*} \, d^2\zeta \\
\sim -\frac{1}{4\pi c_0^2} \frac{\partial}{\partial x_i} \left[ \frac{F_i}{R D} \right]
\]

- In the Fourier domain:

\[
\rho(x, \omega) = \frac{ik}{8\pi^2 c_0^2} \int_{-\infty}^{\infty} \frac{\mathbf{F} \cdot \mathbf{R}}{R^2} \left( 1 + \frac{1}{ikR} \right) e^{-i\omega(\tau + \mathbf{R}/c_0)} \, d\tau
\]
Geometrical far-field approximation for a $B$-bladed axial rotor

Constructive interference:
\[
\text{sound of the total fan} = B \times (\text{sound of a single blade})
\]

Bessel function: modulation of the Doppler frequency shift during blade revolution

\[
\rho_{nB} \sim -\frac{i B k_{nB}}{4\pi c_0^2} \frac{e^{-ik_{nB} x}}{x} \sum_{p=-\infty}^{+\infty} e^{-i(nB-p)(\varphi-\pi/2)} J_{-nB+p}(-k_{nB} R_0 \sin \theta) [F_p^{(T)} \cos \theta - \frac{nB - p}{k_{nB} R_0} F_p^{(D)}]
\]

- **Sound emitted at BPFHs**
- **Sum over BLHs**
- **Listener distance**
- **Listener azimuthal angle**
- **Listener polar angle**
- **Radius where force is applied**
- **Drag harmonic**
- **Thrust harmonic**
Doppler effect $\Rightarrow$ summation over the BLHs

$$B = 4$$

$$\rho_{nB} \sim -\frac{iBk_{nB}}{4\pi c_0^2} \frac{e^{-ik_{nB}x}}{x} \sum_{p=-\infty}^{+\infty} e^{-i(nB-p)(\varphi-\pi/2)} J_{-nB+p}(-k_{nB}R_0 \sin \theta) \left[ F_p^{(T)} \cos \theta - \frac{nB-p}{k_{nB}R_0} F_p^{(D)} \right]$$
For a $B$-bladed rotor downstream of a stator with $V$ vanes:

- Summing up the interferences between the $B$ blades, the **general formulation** yields:

$$\rho_{sB} = \frac{iBk_{sB}}{4\pi c_0^2} \sum_{p=-\infty}^{\infty} \left( - G^{(2)}_{sB-pV} F^{(D)}_{pV} x \sin \theta + G^{(3)}_{sB-pV} F^{(R)}_{pV} x \sin \theta ight)$$

$$+ G^{(1)}_{sB-pV} \left( F^{(T)}_{pV} (\zeta_3 - x \cos \theta) - F^{(R)}_{pV} r' \right)$$

with the auxiliary functions:

$$G_1(t) = \frac{e^{-ikR}}{R^2} \left( 1 + \frac{1}{i k R} \right) \quad G_2(t) = \sin(\Omega t + \varphi' - \varphi) \ G_1(t) \quad G_3(t) = \cos(\Omega t + \varphi' - \varphi) \ G_1(t)$$

- To be compared with the **far-field approximation** (Goldstein, 1976):

$$\rho_{sB} = -\frac{iBk_{sB}}{4\pi c_0^2} \frac{e^{-ik_{sB}x}}{x} \sum_{p=-\infty}^{+\infty} e^{-i(sB-pV)(\varphi'-\pi/2)} \left( J_{-sB+pV}(-k_{sB}r' \sin \theta) F^{(T)}_{pV} \cos \theta \right.$$

$$\left. - \frac{sB - pV}{k_{sB}r'} J_{-sB+pV}(-k_{sB}r' \sin \theta) F^{(D)}_{pV} - i J'_{-sB+pV}(-k_{sB}r' \sin \theta) \sin \theta F^{(R)}_{pV} \right)$$
Example: straight infinite duct

- Assessment of the error based on a comparison with a reference solution
- Reference (exact) solution: acoustic propagation within an infinite cylindrical duct, based on the tailored Green’s function (Goldstein, 1976):

\[
\rho_{sB} = \frac{B}{2c_0^2} \sum_{p=-\infty}^{\infty} \sum_{n=1}^{\infty} \frac{J_m(\kappa_{m,n} r)}{\Gamma_{m,n}} \frac{J_m(\kappa_{m,n} r')}{\kappa_{m,n,sB}} e^{i m (\varphi - \varphi')} e^{i k_{m,n,sB} (z - z')} \left( m F^{(D)}_{PV} \mp k_{m,n,sB} F^{(T)}_{PV} \right)
\]

\[
\Gamma_{m,n} = \pi \left( r_d^2 - \frac{m^2}{\kappa_{m,n}^2} \right) J_m^2(\kappa_{m,n} r_d) \quad k_{m,n,sB} = \sqrt{\left( \frac{\Omega s B}{c_0} \right)^2 - \kappa_{m,n}^2}
\]

- Comparison requires anechoic boundary condition at both duct ends
  - Plane wave impedance \( \rho_0 c_0 \rightarrow \) approximate above cut-off
  - Smooth variation of wall admittance from 0 till \( 1/(\rho_0 c_0) \) over the two last diameters of the duct
  - Locally-conformal Perfectly Matched Layer
Anechoic boundary conditions

- Single dipole inside duct, with non-zero axial, radial and azimuthal components
- Sound spectrum at listener position inside duct
Ducted fan with synthetic blade forces

- Finite Element model: duct with radius $r_d = 1$, length $= 20 r_d + 4 r_d$ for anechoic termination
- Model size 186,000 HEXA8 elements
- Axial fan at origin, synthetic blade forces, $B = 3$, $V = 1$, $\Omega = 1000$ RPM, $r' = 0.2$ m
Analytical and numerical solutions

3rd BPFH

5th BPFH

7th BPFH

tailored (exact)
far-field
near-field
fixed dipoles
Vortex Sound Theory

- Conservation principles
- Powell’s analogy
- Mohring’s analogy
- Conservative formulation
- Application to vortex pairing
Lighthill’s analogy: some issues for free subsonic flows

- Spatial extent of source term for a localized distribution of vorticity (Oseen vortex):
  \[
  v_\theta(r) = \frac{\Gamma}{2\pi r} \left[ 1 - \exp \left( \frac{-r^2}{2\sigma^2} \right) \right]
  \]
  \[
  \omega(r) = \frac{\Gamma}{2\pi \sigma^2} \exp \left( \frac{-r^2}{2\sigma^2} \right)
  \]

- Alternative formulation of the analogy: Vortex Sound Theory
  - Yields a more localised source term
  - Allows the reinforcement of the conservative assumptions
Invariants of incompressible, inviscid vortex flows in absence of external forces

- **Circulation:**
  \[ \Gamma = \oint_C \mathbf{v} \cdot \mathrm{d}l \]
  where \( C \) is a closed material line.

  \[ \frac{d\Gamma}{dt} = \oint_C \mathbf{f} \cdot \mathrm{d}l + \int_C \left( \nabla^2 \mathbf{v} \right) \cdot \mathrm{d}l \]
  vanishes if the force \( f \) derives of a single-valued potential, and for inviscid flows.

- **Impulse (momentum):**
  \[ \mathbf{P} = \frac{1}{2} \rho \iiint_V \mathbf{x} \times \mathbf{\omega} \, \mathrm{d}V \]

  \[ \frac{d\mathbf{P}}{dt} = \iiint_V \mathbf{f} \cdot \frac{1}{\rho} \, \mathrm{d}V \]
  vanishes in absence of non-conservative body forces.

- **Kinetic energy:**
  \[ T = \rho \iiint_V \mathbf{v} \cdot (\mathbf{x} \times \mathbf{\omega}) \, \mathrm{d}V \]
  conserved quantity using the same assumptions.
Vortex Sound Theory: Powell’s analogy for free flows

- Vectorial identity: \( \nabla \left( \frac{|\mathbf{v}|^2}{2} \right) = \mathbf{v} \times \mathbf{\omega} + \mathbf{v} \cdot \nabla \mathbf{v} \)

- Momentum equation becomes: \( \rho \frac{\partial \mathbf{v}}{\partial t} + \rho \nabla \left( \frac{1}{2} |\mathbf{v}|^2 \right) + \rho (\mathbf{\omega} \times \mathbf{v}) + \nabla p = 0 \)

- Similar manipulation as for Lighthill’s analogy:

  \[
  \frac{1}{c_0^2} \frac{\partial^2 p'}{\partial t^2} - \frac{\partial^2 p'}{\partial x_i^2} = \nabla \cdot \left[ \rho (\mathbf{\omega} \times \mathbf{v}) + \nabla \left( \frac{1}{2} \rho |\mathbf{v}|^2 \right) \right] - \mathbf{v} \cdot \frac{\partial \rho}{\partial t} - \frac{1}{2} |\mathbf{v}|^2 \nabla \rho + \frac{\partial}{\partial t^2} \left( \frac{p'}{c_0^2} - \rho' \right) \]

  \( \propto M^2 \)

- Retaining leading order terms in \( M^2 \):

  \[
  \frac{1}{c_0^2} \frac{\partial^2 p'}{\partial t^2} - \frac{\partial^2 p'}{\partial x_i^2} = \nabla \cdot \left[ \rho (\mathbf{\omega} \times \mathbf{v}) \right] + \nabla^2 \left( \frac{1}{2} \rho |\mathbf{v}|^2 \right) \]

- Integral solution using free field Green’s function and first order Taylor expansion of the retarded time:

  \[
  p'(\mathbf{x}, t) = -\frac{x_i}{4\pi c_0 |\mathbf{x}|^2} \frac{\partial}{\partial t} \int \int \rho(\mathbf{\omega} \times \mathbf{v}) \, d^3y - \frac{x_i x_j}{4\pi c_0^2 |\mathbf{x}|^3} \frac{\partial^2}{\partial t^2} \int V y_j \rho(\mathbf{\omega} \times \mathbf{v}) \, d^3y
  \]

- Conservation of impulse

- Powell’s integral formulation:

  \[
  p'_P(\mathbf{x}, t) = -\frac{\rho_0}{4\pi c_0^2 |\mathbf{x}|^3} \frac{\partial^2}{\partial t^2} \int \int V (\mathbf{x} \cdot \mathbf{y}) \cdot (\mathbf{\omega} \times \mathbf{v}) \, d^3y
  \]
Vortex Sound Theory: Möhring’s analogy for free flows

- Starting from Powell’s integral formulation:

\[ p'_P(x, t) = -\frac{\rho_0}{4\pi c_0^2 |x|^3} \frac{\partial^2}{\partial t^2} \iiint_V (x \cdot y) x \cdot (\omega \times v) \, d^3y \]

- Using vectorial’s identity:

\[ \nabla_y \times \left[ \frac{1}{3} (x \cdot y) x \times y \right] = (x \cdot y) x - \frac{1}{3} |x|^2 y \]

- By substitution:

\[ p'(x, t) = -\frac{\rho_0}{12\pi c_0^2 |x|^3} \frac{\partial^2}{\partial t^2} \left\{ \iiint_V \nabla_y \times [(x \cdot y) x \times y] \cdot (\omega \times v) \, d^3y \right\} \]

\[ + |x|^2 \iiint_V \cdot (\omega \times v) \, d^3y \]

conservation of kinetic energy

- Using Helmholtz’s vorticity transport equation:

\[ \frac{\partial \omega}{\partial t} + \nabla_y \times (\omega \times v) = 0 \]

- Möhring’s integral formulation:

\[ p'_M(x, t) = \frac{\rho_0}{12\pi c_0^2 |x|^3} \frac{\partial^3}{\partial t^3} \iiint_V (x \cdot y) x \cdot (y \times \omega) \, d^3y \]
Vortex Sound Theory: 2 solutions for the same problem

- We have derived two (formally) equivalent formulations of the Vortex Sound Theory:

  - Powell’s analogy:  
    \[ p'_P(x, t) = -\frac{\rho_0}{4\pi c_0^2|x|^3} \frac{\partial^2}{\partial t^2} \iiint_V (x \cdot y) x \cdot (\omega \times v) \, d^3y \]

  - Mohring’s analogy:  
    \[ p'_M(x, t) = \frac{\rho_0}{12\pi c_0^2|x|^3} \frac{\partial^3}{\partial t^3} \iiint_V (x \cdot y) x \cdot (y \times \omega) \, d^3y \]

- Although formally equivalent, these two formulations do not yield the same numerical robustness!

  The choice of a source term affects the numerical performance of the prediction!
Vortex Sound Theory for axisymmetrical flows

- Coordinate of a vortex element: \( y = z \mathbf{n} + r \mathbf{e}(\phi) \)

- General form of velocity and vorticity:

\[
\begin{align*}
\mathbf{\omega}(r, \phi, z) &= \omega(r, z) \mathbf{n} \times \mathbf{e}(\phi) \\
\mathbf{v}(r, \phi, z) &= u(r, z) \mathbf{n} + v(r, z) \mathbf{e}(\phi) + w(r, z) \mathbf{n} \times \mathbf{e}(\phi)
\end{align*}
\]

- Powell’s analogy becomes:

\[
p'_p(x, t) = \frac{\rho_0}{4c_0^2|x|^3} \frac{\partial^2}{\partial t^2} \left\{ 2 \left( \int_S \omega r z \, dr \, dz \right) (x \cdot n)(x \cdot n) - \left( \int_S \omega r u^2 \, dr \, dz \right) [(x \cdot x) - (x \cdot n)(x \cdot n)] \right\}
\]

- Möhring’s analogy becomes:

\[
p'_M(x, t) = \frac{\rho_0}{4c_0^2|x|^3} \frac{d^3Q}{dt^3} x \cdot \left( \mathbf{n} n - \frac{1}{3} \right) \cdot x
\]

\[
Q = \int_S \omega r^2 z \, dr \, dz
\]
Application: vortex ring pairing

- Vortex pairing = inviscid interaction (Biot-Savart)
  - Vortex leapfrogging: periodic motion
  - Vortex merging: requires core deformation
- Can be easily stabilized and studied at laboratory scale
- One of the mechanisms of sound production in subsonic jets
- Generic interaction showing how the reciprocal exchange of impulse $|b|$ two vortex elements produces a quadrupole in far field

$$\mathbf{P} = \rho_0 \Gamma \mathbf{A} \quad \mathbf{A} = \pi R^2 \mathbf{n}$$

$$p'(\mathbf{x}, t) = \frac{\rho_0}{4\pi c_0 |\mathbf{x}|} \frac{d^2}{dt^2} (\Gamma A_x)_{t-|\mathbf{x}|/c_0} \quad \text{for each vortex ring}$$

$$\frac{d}{c_0} \ll \frac{d}{U_0} \quad M = \frac{U_0}{c_0} \ll 1 \quad \frac{d}{\lambda} \simeq \frac{d}{c_0} \frac{U_0}{d} = M$$
Vortex pairing: $U_0 = 5.0 \text{ m/s}$
2D and 3D models of vortex ring leapfrogging

- **2D model** ($\sigma \ll d \ll R_0$): locally planar interaction, neglects vortex stretching:

\[
\begin{align*}
Z_L(t) &= u_0 t + \frac{d}{2} \cos \left( \frac{\Gamma t}{\pi d^2} \right) \\
R_L(t) &= R_0 + \frac{d}{2} \sin \left( \frac{\Gamma t}{\pi d^2} \right) \\
Z_T(t) &= u_0 t - \frac{d}{2} \cos \left( \frac{\Gamma t}{\pi d^2} \right) \\
R_T(t) &= R_0 - \frac{d}{2} \sin \left( \frac{\Gamma t}{\pi d^2} \right)
\end{align*}
\]

- **3D model** ($\sigma \ll d = O(R_0)$): accounts for vortex stretching:

\[
\begin{align*}
\frac{dZ_m}{dt} &= \frac{1}{R_m} \frac{\partial \Psi}{\partial R_m} + \frac{\Gamma_m}{4\pi R_m} \left[ \log \left( \frac{8R_m}{\sigma_{c,m}} \right) - \frac{1}{4} \right] \\
\frac{dR_m}{dt} &= -\frac{1}{R_m} \frac{\partial \Psi}{\partial Z_m} \\
\frac{d\sigma_{c,m}^2 R_m}{dt} &= 0 \\
\Psi &= \frac{\Gamma_n}{2\pi} \sqrt{R_m R_n} \left[ \left( \frac{2}{k_{mn}} - k_{mn} \right) K(k_{mn}) - \frac{2}{k_{mn}} E(k_{mn}) \right]
\end{align*}
\]

\[
k_{mn} = \sqrt{\frac{4R_m R_n}{(Z_m - Z_n)^2 + (R_m + R_n)^2}}
\]
2D model: vortex trajectories and flow invariants

- Two cases considered: $d / R_0 = 0.1$ and $0.3$
- Locus of the vortex cores:

![Diagram showing vortex trajectories and flow invariants](image)

- Flow invariants:

$$P = 2\pi \rho_0 \Gamma R_0^2 \left[ 1 + \frac{d^2}{4R_0^2} \sin^2 \left( \frac{\Gamma t}{\pi d^2} \right) \right]$$

$$T = 2\pi \rho_0 \Gamma \left[ 2R_0^2 u_0 - \frac{\Gamma R_0}{2\pi} + \left( \frac{u_0 d^2}{2} - \frac{\Gamma R_0}{2\pi} \right) \sin^2 \left( \frac{\Gamma t}{\pi d^2} \right) - \frac{u_0 \Gamma}{4\pi} t \sin \left( \frac{2\Gamma t}{\pi d^2} \right) \right] \text{ secular term}$$
2D model: sound prediction

- **Powell’s analogy:**

\[
p'_{D}(x, t) = \frac{\rho_0}{4c_0^2|x|^3} \left\{ \left[ \frac{-4\Gamma^4 R_0}{\pi^3 d^4} + \frac{3\Gamma^4 u_0}{\pi^2 d^2} \right] \cos \left( \frac{2\Gamma t}{\pi^2 d^2} \right) \\
- \frac{2\Gamma^4 u_0}{\pi^3 d^4} t \sin \left( \frac{2\Gamma t}{\pi^2 d^2} \right) \right\} x \cdot (n \cdot n) \cdot x
\]

- **Möhring’s analogy:**

\[
p'_{M}(x, t) = \frac{\rho_0}{4c_0^2|x|^3} \left\{ \frac{3\Gamma^2 u_0}{\pi^2 d^2} - \frac{4\Gamma^4 R_0}{\pi^3 d^4} \right\} \cos \left( \frac{2\Gamma t}{\pi^2 d^2} \right) \\
- \frac{2\Gamma^4 u_0}{\pi^3 d^4} t \sin \left( \frac{2\Gamma t}{\pi^2 d^2} \right) \right\} x \cdot (n \cdot n - \frac{1}{3}) \cdot x
\]

- **Conclusion:** failure of both Powell’s and Möhring’s analogies when applied to a flow model that does not respect the conservation of momentum and kinetic energy.
Möhring’s solution: reinforcement of conservation assumptions

- Using Lamb (1932) identities:

\[
\frac{dQ}{dt} = \frac{T}{2\pi \rho_0} + 3\Gamma (R_L V_L Z_L + R_T V_T Z_T)
\]

- Imposing further conservation of impulse:

\[
\frac{d}{dt} \left( R_L^2 + R_T^2 \right) = 0 \quad R_L V_L = -R_T V_T
\]

- We obtain:

\[
\frac{dQ}{dt} = \frac{T}{2\pi \rho_0} + 3\Gamma R_L V_L (Z_L - Z_T)
\]

- Imposing further conservation of kinetic energy:

\[
p'(x, t) = \frac{-3}{4} \frac{\rho_0 \Gamma^4 R_0}{\pi^3 d^4 c_0^2 |x|^3} \cos \left( \frac{2\Gamma t}{\pi d^2} \right) x \cdot \left( nn - \frac{I}{3} \right) \cdot x
\]
Generalization of Möhring’s solution

- Using Lamb (1932) identities:
  \[ \frac{dQ}{dt} = \frac{T}{2\pi \rho_0} + 3 \int_S \omega vr \, dr \, dz \]
  will disappear in subsequent derivations

- Second correction: subtraction of the vortex centroid axial coord. from the axial coord. of each vortex element.

\[ z_0 = \frac{\int_S \omega r^2 \, z \, dr \, dz}{\int_S \omega r^2 \, dr \, dz} \]

- Doesn’t harm if impulse is conserved, since

\[ \int_S \omega vr \, dr \, dz = \frac{1}{2\pi \rho_0} \frac{dP}{dt} \]

- Improves numerical stability.

- Imposing further conservation of impulse and kinetic energy:

\[ p'_{c,0}(\mathbf{x}, t) = \frac{3\rho_0}{4c_0^2 |\mathbf{x}|^3} \frac{d^2}{dt^2} \left( \int_S \omega r (z - z_0) \, dr \, dz \right) \mathbf{x} \cdot \left( \mathbf{n} \, \mathbf{n} - \frac{1}{3} \right) \cdot \mathbf{x} \quad \text{conservative formulation} \]
Application to PIV data
PIV results: flow invariants

- Low frequency fluctuations of about 10%.
- Increasing scatter due to growing instabilities.
PIV results: acoustical source terms

\[
\int \int_S \omega r^2 z \, drdz
\]

\[
\frac{T}{2\pi \rho_0} + 3 \int \int_S \omega vr (z - z_0) \, drdz
\]

\[
3 \int \int_S \omega vr (z - z_0) \, drdz
\]
Acoustic predictions

- 2nd time derivative: 4th order polynomial fit over moving interval
  ➔ acoustical source term.

- Case $U_0 = 5.0 \text{ m/s}$: good agreement between predictions obtained from PIV data and 3D leapfrogging model.

- Case $U_0 = 34.2 \text{ m/s}$: order of magnitude OK, but quite different frequency content.
Surprise: double pairing
Intermittence of double pairing and comparison prediction - measurement

- For some acquisitions: absence of double pairing.
- Quite good agreement between prediction and measurement, for first and third pairing harmonics.
Summary

- Aeroacoustical analogies allow extracting a maximum of acoustical information from a given description of the flow field.

- Assuming a decoupling between the sound production and propagation, the analogies provide an explicit integral solution for the acoustical field at the listener position.
  - Improves numerical robustness
  - Permits drawing scaling laws

- BUT: one has to make approximations and choices!
  - Acoustical variable (e.g. isothermal noise vs combustion noise)
  - Source term formulation (e.g. Lighthill’s analogy vs Vortex Sound Theory)

- Some formulations make the dominant character of the source appear more explicitly, and allow making useful approximations.
- Without approximations, the analogy is useless!
A few references

- S.W. Rienstra and A. Hirschberg, *An Introduction To Acoustics* (corrections), Report IWDE 01-03 May 2001, revision every year or so...

- And of course: the VKI Lecture Series...